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Scattering Solutions in Networks of Thin Fibers: Small Diameter Asymptotics.

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Abstract

Small diameter asymptotics is obtained for scattering solutions in a network of thin fibers. The asymptotics is expressed in terms of solutions of related problems on the limiting quantum graph Γ. We calculate the Lagrangian gluing conditions at vertices $v\in\Gamma$ for the problems on the limiting graph. If the frequency of the incident wave is above the bottom of the absolutely continuous spectrum, the gluing conditions are formulated in terms of the scattering data for each individual junction of the network.

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1 Formulation of the problem and statement of the results

The paper concerns the asymptotic analysis of wave propagation through a system of wave guides when the thickness ε of the wave guides is very small and the wave length is comparable to ε. The problem is described by the stationary wave (Helmholtz) equation

$$-\varepsilon^2\Delta u=\lambda u, \qquad x\in\Omega_\varepsilon, \tag{1}$$

in a domain $\Omega_\varepsilon\subset R^d$, $d\geq 2$, with infinitely smooth boundary (for simplicity) which has the following structure: Ω_ε is a union of a finite number of cylinders $C_{j,\varepsilon}$ (which we shall call channels), $1\leq j\leq N$, of lengths l_j with the diameters of cross-sections of order $O(\varepsilon)$ and domains $J_{1,\varepsilon},\cdots,J_{M,\varepsilon}$ (which we shall call junctions) connecting the channels into a network. It is assumed that the junctions have diameters of the same order $O(\varepsilon)$. Let m channels have infinite length. We start the numeration of $C_{j,\varepsilon}$ with the infinite channels. So, $l_j=\infty$ for $1\leq j\leq m$. The axes of the channels form edges Γ_j of the limiting ($\varepsilon\to 0$) metric graph Γ. The vertices $v_j\in V$ of the graph Γ correspond to the junctions $J_{j,\varepsilon}$.

The Helmholtz equation in Ω_ε must be complemented by the boundary conditions (BC) on $\partial\Omega_\varepsilon$. In some cases (for instance, when studying heat transport in Ω_ε) the Neumann BC is natural. In fact, the Neumann BC presents the simplest case due to the existence of a simple ground state (a constant) of the problem in Ω_ε. However, in many applications, the Dirichlet, Robin or impedance BC are more important. We shall consider (apart from a general discussion) only the Dirichlet BC, but all the arguments and results can be modified to be applied to the problem with other BC.

An important class of domains Ω_ε are self-similar domains with only one junction and all the channels being infinite. We will call them *spider domains*. Thus, if Ω_ε is a spider domain, then there exist a point $\widehat{x}=x(\varepsilon)$ and an ε-independent domain Ω such that

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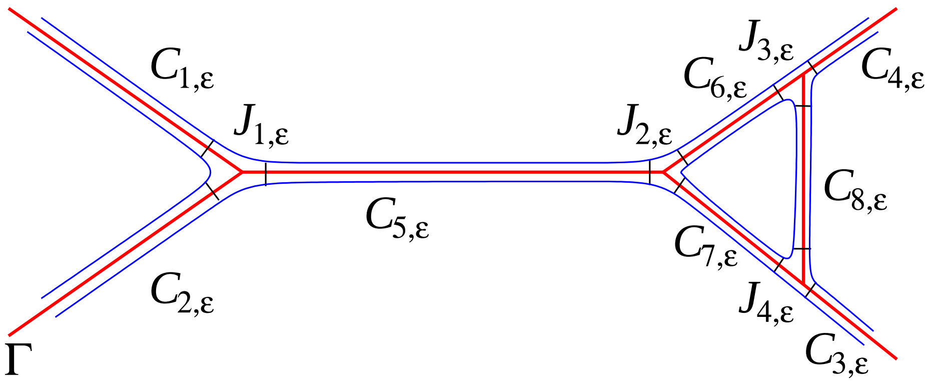

Figure 1: An example of a domain Ω_ε with four junctions, four unbounded channels and four bounded channels.

$$\Omega_\varepsilon = \{(\widehat{x} + \varepsilon x) : x \in \Omega\}. \tag{2}$$

Thus, Ω_ε is ε-contraction of $\Omega = \Omega_1$.

For any Ω_ε, let $J_{j(v),\varepsilon}$ be the junction which corresponds to a vertex $v \in V$ of the limiting graph Γ. Consider a junction $J_{j(v),\varepsilon}$ and all adjacent to $J_{j(v),\varepsilon}$ channels. If some of these channels have a finite length, we extend them to infinity. We assume that, for each $v \in V$, the resulting domain $\Omega_{v,\varepsilon}$ which consists of junction $J_{j(v),\varepsilon}$ and emanating from it semi-infinite channels is a spider domain (i.e., $\Omega_{v,\varepsilon}$ is self-similar). This assumption can be weakened. For example, one can consider some type of "curved" channels, and the final results (with some changes) will remain valid. Simple equations on the limiting graph in this case will be replaced by more complicated equations with variable coefficients. However, even small deviation from the assumption on the self-similarity of $\Omega_{v,\varepsilon}$ would make the statement of the results and the proofs much more technical. So, we consider only domains Ω_ε for which $\Omega_{v,\varepsilon}$, $v \in V$, are self-similar.

Hence, the cross sections $\omega_{j,\varepsilon}$ of channels $C_{j,\varepsilon}$ are $\varepsilon-$homothety of bounded domains $\omega_j \in R^{d-1}$. Let $\lambda_{j,0} < \lambda_{j,1} \leq \lambda_{j,2}...$ be eigenvalues of the negative Laplacian $-\Delta_{d-1}$ in ω_j with the Dirichlet boundary condition on $\partial\omega_j$, and let $\{\varphi_{j,n}\}$ be the set of corresponding orthonormal eigenfunctions. The eigenvalues $\lambda_{j,n}$ coincide with the eigenvalues of $-\varepsilon^2\Delta_{d-1}$ in $\omega_{j,\varepsilon}$. In the presence of infinite channels, the spectrum of the operator $-\varepsilon^2\Delta$ in Ω_ε with the Dirichlet boundary condition on $\partial\Omega_\varepsilon$ has an absolutely continuous component which coincides with the semi-bounded interval $[\lambda_0, \infty)$, where

$$\lambda_0 = \min_{1\leq j\leq m} \lambda_{j,0} \tag{3}$$

The equation (1) is considered under the assumption that $\lambda \geq \lambda_0$, when propagation of waves is possible. There are two very different cases: $\lambda \to \lambda_0$ as $\varepsilon \to 0$, i.e. the frequency is at the edge (or bottom) of the absolutely continuous spectrum, or $\lambda \to \widehat{\lambda} > \lambda_0$, i.e. the frequency is above the bottom of the absolutely continuous spectrum. There are many results about the first case, the references will be given later. This paper concerns the asymptotic analysis of the scattering solutions for the Dirichlet problem in Ω_ε when λ is close to $\widehat{\lambda} > \lambda_0$.

If $\varepsilon \to 0$, one can expect that the solution u_ε of (1) in Ω_ε can be described in terms of the solution $\varsigma = \varsigma_\varepsilon(t)$ of a much simpler problem on the graph Γ. For example, if $\lambda_{j,0} < \lambda < \lambda_{j,1}$ for all j, then ς satisfies the following equation on each edge of the graph

$$-\frac{\varepsilon^2 d^2\varsigma\,(t)}{dt^2} = (\lambda - \lambda_{j,0})\varsigma\,(t)\,, \tag{4}$$

where t is the length parameter on the edges. One has to add appropriate gluing conditions (GC) at the vertices v of Γ. These gluing conditions give basic information on the propagation of waves through the junctions. They define the solution ς of the problem (4) on the limiting graph. The ordinary differential equation (4), the GC, and the solution ς depend on ε. However, we shall often call the corresponding problem on the graph the limiting problem, since it enables one to find the main term of the asymptotics as $\varepsilon \to 0$ for the solution $u = u_\varepsilon$ of the problem (1) in Ω_ε.

One of the main difficulties in the problem under investigation was to find the GC, in particular, since the GC differ dramatically from those which were known in the case of λ close to the bottom of the spectrum.

Let us define the scattering solutions for the Dirichlet problem in Ω_ε. We introduce local coordinates (t, y) in each channel $C_{j,\varepsilon}$ with t axis parallel to the cylinder $C_{j,\varepsilon}$, $0 < t < l_j$, and $y \in R^{n-1}$ being Euclidean coordinates in the plane perpendicular to the t axis. The coordinate y is chosen in such a way that $\omega_{j,\varepsilon} = \{(\varepsilon y) : y \in \omega_j \in R^{n-1}\}$. For each j, the set $\{\varepsilon^{\frac{1-d}{2}}\varphi_{j,n}(\frac{y}{\varepsilon})\}$ is the orthonormal basis in $L^2(\omega_{j,\varepsilon})$ consisting of eigenfunctions of the operator $-\varepsilon^2\Delta_{d-1}$.

Let l be a bounded closed interval of the real axis which does not contain the points $\lambda_{j,n}$, $j \leq N$. Thus, there exist $m_j \geq 1$ such that $\lambda_{j,m_j} < \lambda < \lambda_{j,m_j+1}$ for all $\lambda \in l$. As will be seen from the definitions below, $m_j + 1$ is the number of waves which may propagate in each direction in the channel $C_{j,\varepsilon}$ without loss of energy and with frequencies less than $\sqrt{\lambda}$, $\lambda \in l$. We put $m_j = -1$, thus $\{\lambda_{j,n}, 0 \leq n \leq m_j\}$ is the empty set if $\lambda_{j,0} > \lambda$ for $\lambda \in l$.

Consider the non-homogeneous Dirichlet problem

$$(-\varepsilon^2\Delta - \lambda)u = f, \quad x \in \Omega_\varepsilon; \qquad u = 0 \;\text{ on } \partial\Omega_\varepsilon. \tag{5}$$

Definition 1 *Let $f \in L^2_{com}(\Omega_\varepsilon)$ have a compact support, and $\lambda \in l$. A solution u of (5) is called an outgoing solution if it has the following asymptotic behavior at infinity in each infinite channel $C_{j,\varepsilon}$, $1 \leq j \leq m$:*

$$u = \sum_{n=0}^{m_j} a_{j,n} e^{i\frac{\sqrt{\lambda-\lambda_{j,n}}}{\varepsilon}t}\varphi_{j,n}(y/\varepsilon) + O(e^{-\gamma t}), \quad \gamma = \gamma(\varepsilon) > 0, \tag{6}$$

Definition 2 *A function $\Psi = \Psi^{(\varepsilon)}_{s,k}$, $1 \leq s \leq m$, $0 \leq k \leq m_j$, is called a solution of the scattering problem in Ω_ε if*

$$(-\varepsilon^2\Delta - \lambda)\Psi = 0, \quad x \in \Omega_\varepsilon; \qquad \Psi = 0 \;\text{ on } \partial\Omega_\varepsilon, \tag{7}$$

and Ψ has the following asymptotic behavior at infinity in each infinite channel $C_{j,\varepsilon}$, $1 \leq j \leq m$:

$$\Psi^{(\varepsilon)}_{s,k} = \delta_{s,j} e^{-i\frac{\sqrt{\lambda-\lambda_{s,k}}}{\varepsilon}t}\varphi_{s,k}(y/\varepsilon) + \sum_{n=0}^{m_j} t_{j,n} e^{i\frac{\sqrt{\lambda-\lambda_{j,n}}}{\varepsilon}t}\varphi_{j,n}(y/\varepsilon) + O(e^{-\gamma t}), \tag{8}$$

where $\gamma = \gamma(\varepsilon) > 0$, and $\delta_{s,j}$ is the Kronecker symbol, i.e. $\delta_{s,j} = 1$ if $s = j$, $\delta_{s,j} = 0$ if $s \neq j$.

The first term in (8) corresponds to the incident wave, and all other terms describe the transmitted waves. The incident wave depends on s and k, where s determines the channel, and s and k together determine the frequency of the incident wave. The transmission coefficients $t_{j,n}$ also depend on s and k (i.e. on the choice of the incident wave), so sometimes we will denote them by $t^{s,k}_{j,n}$.

We introduce an order in the set of incident waves and corresponding scattering solutions and the same order in the set of transmitted waves. Namely, we number the incident waves in the channel $C_{1,\varepsilon}$ taking them in the order of increase of absolute values of their frequencies, then we number all the solutions in the channel $C_{2,\varepsilon}$, and so on. With this order taken into account, the transmission coefficients for a particular scattering solution form a column vector with

$$M = \sum_{j=1}^{m}(m_j + 1) \tag{9}$$

entries. Together, they form an $M \times M$ *scattering matrix*

$$T = \{t_{j,n}^{s,k}\}, \tag{10}$$

where s, k define the column of T and j, n define the row. We denote by D the diagonal $M \times M$ matrix with elements $\sqrt{\lambda - \lambda_{j,n}}$ on the diagonal taken in the same order as above. The following statement can be useful in some applications, and will be proved in the next section (although it will not be used in this paper).

Theorem 3 *The matrix $D^{1/2}TD^{-1/2}$ is unitary and symmetric.*

The operator $H = -\varepsilon^2\Delta$ with the Dirichlet boundary conditions on $\partial\Omega_\varepsilon$ is non-negative, and therefore the resolvent

$$R_\lambda = (-\varepsilon^2\Delta - \lambda)^{-1} : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon) \tag{11}$$

is analytic in the complex λ plane outside the positive semi-axis $\lambda \geq 0$. Hence, the operator R_{k^2} is analytic in k in the half plane Im$k > 0$. We are going to consider an analytic extension of the operator R_{k^2} onto the real axis and in the lower half plane. Such an extension does not exist if R_{k^2} is considered as an operator in $L^2(\Omega_\varepsilon)$ since R_{k^2} is an unbounded operator when $\lambda = k^2$ belongs to the spectrum of the operator R_λ. However, one can extend R_{k^2} analytically if it is considered as an operator in the following spaces (with a smaller domain and a larger range):

$$R_{k^2} : L^2_{com}(\Omega_\varepsilon) \to L^2_{loc}(\Omega_\varepsilon). \tag{12}$$

Theorem 4 *(1) The spectrum of the operator $H = -\varepsilon^2\Delta$ in Ω_ε with the Dirichlet boundary conditions on $\partial\Omega_\varepsilon$ consists of the absolutely continuous component $[\lambda_0, \infty)$ where $\lambda_0 > 0$ is given by (3) and, possibly, a discrete set of positive eigenvalues $\{\lambda^{j,\varepsilon}\}$ with the only possible limiting point at infinity. The multiplicity of the a.c. spectrum changes at points $\lambda = \lambda_{j,n}$, and at any point λ, it is equal to the number of points $\lambda_{j,n}$, $1 \leq j \leq m$, located below λ. The eigenvalues $\lambda^{j,\varepsilon} = \lambda^j$ for spider domains Ω_ε do not depend on ε.*

(2) The operator (12) admits a meromorphic extension from the upper half plane Im$k > 0$ into lower half plane Im$k < 0$ with the branch points at $k = \pm\sqrt{\lambda_{j,n}}$ of the second order and the real poles at $k = \pm\sqrt{\lambda^{s,\varepsilon}}$ and, perhaps, at some of the branch points. The resolvent (12) has a pole at $k = \pm\sqrt{\lambda_{j,n}}$ if and only if the homogeneous problem (5) with $\lambda = \lambda_{j,n}$ has a nontrivial solution u such that

$$u = \sum_{j,n:\lambda_{j,n}=\lambda} a_{j,n}\varphi_{j,n}(y/\varepsilon) + (e^{-\gamma t}), \qquad x \in C_{j,\varepsilon}, \quad t \to \infty, \quad 1 \leq j \leq m. \tag{13}$$

(3) If $f \in L^2_{com}(\Omega_\varepsilon)$, and $k = \sqrt{\lambda}$ is real and is not a pole or a branch point of the operator (12), and $\lambda > \lambda_0$, then the problem (5), (6) is uniquely solvable and the outgoing solution u can be found as the $L^2_{loc}(\Omega_\varepsilon)$ limit

$$u = R_{\lambda+i0}f. \tag{14}$$

(4) There exist exactly M (see (9)) different scattering solutions for values of $\lambda > \lambda_0$ such that $k = \sqrt{\lambda}$ is not a pole or a branch point of the operator (12), and the scattering solution is defined uniquely after the incident wave is chosen.

Remarks. 1. Operator $H = -\varepsilon^2\Delta$ and its domain depend on ε. One could use the term "family of operators" when referring to H. We prefer to drop the word "family", but one must always keep in mind that H depends on ε.

2. Existence of a pole of the operator (12) at a branch point means that R_{k^2} has a pole at $z = 0$ if this operator function is considered as a function of $z = \sqrt{k^2 - \lambda_{j,n}}$.

3. One can not identify poles of the resolvent and eigenvalues of the operator based only on general theorems of functional analysis since we deal with the poles of the modified resolvent (12) which belong to the absolutely continuous spectrum of the operator.

4. The eigenvalues $\lambda^{j,\varepsilon}$ of the operator H can be embedded into the absolutely continuous spectrum, and can be located below the absolutely continuous spectrum. In particular, from the minimax principle it follows that H necessarily has a non-empty discrete spectrum below λ_0 if at least one of the junctions is wide enough. For example, non-empty discrete spectrum below λ_0 exists if a junction contains a ball B_ρ of the radius $\rho = r\varepsilon$ such that the negative Dirichlet Laplacian in the ball B_r has an eigenvalue below λ_0.

Let us describe the asymptotic behavior of scattering solutions $\Psi = \Psi^{(\varepsilon)}_{s,k}$ as $\varepsilon \to 0$, $\lambda \in l$. Note that an arbitrary solution u of the equation (1) in a channel $C_{j,\varepsilon}$ can be represented as a series with respect to the orthogonal basis $\{\varphi_{j,n}(y/\varepsilon)\}$ of the eigenfunctions of the Laplacian in the cross-section of $C_{j,\varepsilon}$. Thus it can be represented as a linear combination of the travelling waves

$$e^{\pm i\frac{\sqrt{\lambda-\lambda_{j.n}}}{\varepsilon}t}\varphi_{j,n}(y/\varepsilon), \quad 1 \leq n \leq m_j,$$

and functions which grow or decay exponentially along the axis of $C_{j,\varepsilon}$. The main term of small ε asymptotics of scattering solutions contains only travelling waves, i.e. on each channel $C_{j,\varepsilon}$, any function Ψ has the form

$$\Psi = \Psi^{(\varepsilon)}_{s,k} = \sum_{n=0}^{m_j}(\alpha_{j,n}e^{i\frac{\sqrt{\lambda-\lambda_{j.n}}}{\varepsilon}t} + \beta_{j,n}e^{-i\frac{\sqrt{\lambda-\lambda_{j.n}}}{\varepsilon}t})\varphi_{j,n}(y/\varepsilon) + r^{\varepsilon}_{s,k}, \tag{15}$$

where

$$|r^{\varepsilon}_{s,k}| \leq Ce^{-\frac{\gamma d(t)}{\varepsilon}}, \quad \gamma > 0, \quad \text{and} \quad d(t) = \min(t, l_j - t).$$

The constants $\alpha_{j,n}$ and $\beta_{j,n}$ depend also on s, k and ε. The formula (15) can be written in a shorter form as follows

$$\Psi = \Psi^{(\varepsilon)}_{s,k} = \sum_{n=0}^{m_j}\varsigma_j \cdot \varphi_j + r^{\varepsilon}_{s,k}, \qquad |r^{\varepsilon}_{s,k}| \leq Ce^{-\frac{\gamma d(t)}{\varepsilon}},$$

where $\varphi_j = \varphi_j(y/\varepsilon)$ is the vector with components $\varphi_{j,n}(y/\varepsilon)$, $0 \leq n \leq m_j$, and $\varsigma_j = \varsigma_j(t)$ is a (m_j+1)-vector whose components $\varsigma_{j,n}$ are linear combinations of the corresponding oscillating exponents in t, i.e. ς_j satisfies the following equation:

$$(\varepsilon^2\frac{d^2}{dt^2} + D_j^2)\varsigma_j = 0, \quad 0 < t < l_j, \tag{16}$$

where D_j is the diagonal matrix with elements $\sqrt{\lambda - \lambda_{j.n}}$, $0 \leq n \leq m_j$, on the diagonal.

In order to complete the description of the main term of the asymptotic expansion (15), we need to provide the choice of constants in the representation of $\varsigma_{j,n}$ as linear combinations of the exponents. Thus, $2(m_j+1)$ constants must be chosen for each channel $C_{j,\varepsilon}$. We consider the limiting graph Γ, whose edges Γ_j are the axes of the channels $C_{j,\varepsilon}$. Let ς be the vector valued function on Γ which is equal to ς_j on Γ_j. The vector ς has a different number of coordinates on different edges Γ_j of the graph Γ. We specify ς by imposing conditions at infinity and gluing conditions (GC) at each vertex v of the graph Γ. Let $V = \{v\}$ be the set of vertices v of the limiting graph Γ. These vertices correspond to the junctions in Ω_ε.

The conditions at infinity concern only the infinite channel $C_{j,\varepsilon}$, $j \leq m$. They depend on the choice of the incident wave and have the form:

$$\beta_{j,n} = \begin{cases} 1 \text{ if } & (j,n) = (s,k) \\ 0 \text{ if } & (j,n) \neq (s,k) \end{cases}, \quad 1 \leq j \leq m. \tag{17}$$

The GC at vertices v of the graph Γ are universal for all incident waves and depend on λ. In order to state the GC at a vertex v , we choose the parametrization on Γ in such a way that $t = 0$ at v for all edges adjacent to this particular vertex. The origin ($t = 0$) on all other edges can be chosen at any of the end points of the edge. Consider auxiliary scattering problems for the spider type domain $\Omega_{v,\varepsilon}$ formed by the individual junction, which corresponds to the vertex v, and all channels with an end at this junction, where the channels are extended to infinity if they have a finite length. We denote by Γ_v the limiting graph which is defined by $\Omega_{v,\varepsilon}$. Definitions 1, 2 and Theorem 4 remain valid for the domain $\Omega_{v,\varepsilon}$. In particular, one can define the scattering matrix $T = T_v$ for the problem (1) in the domain $\Omega_{v,\varepsilon}$. Let v_1, v_2, $...v_l$, $l = l(v)$, be indices of channels in Ω_ε which correspond to channels in $\Omega_{v,\varepsilon}$. Les us form a vector $\varsigma^{(v)}$ by writing the coordinates of all vectors ς_{v_s} in one column, starting with coordinates of ς_{v_1}, then coordinates of ς_{v_2}, and so on. Let us denote by $D_v(\lambda)$ the diagonal matrix with the diagonal elements $\sqrt{\lambda - \lambda_{v_{s,k}}}$ written in the same order as the coordinates of the vector $\varsigma^{(v)}$. Let I_v be the unit matrix of the same size as the size of the matrix $D_v(\lambda)$. The GC at the vertex v has the form

$$\varepsilon[I_v + T_v]D_v^{-1}(\lambda)\frac{d}{dt}\varsigma^{(v)}(t) + i[I_v - T_v]\varsigma^{(v)}(t) = 0, \qquad t = 0. \tag{18}$$

The GC (18) has the following form in the coordinate representation. Let $Z = Z(v)$ be the set of indices (j, n), where j are the indices of the edges of Γ ending at v and $0 \le n \le m_j$. Then

$$\sum_{(j,n)\in Z} \left\{ \varepsilon \left[\delta_{j,n}^{s,k} + t_{j,n}^{s,k}(v)\right] (\lambda - \lambda_{j,n})^{-1/2} \frac{d}{dt}\varsigma_{j,n} + i \left[\delta_{j,n}^{s,k} - t_{j,n}^{s,k}(v)\right] \varsigma_{j,n} \right\} \;=\; 0 \;\text{ at }\; v, \qquad (s,k) \;\in\; Z,$$

where $t_{j,n}^{s,k}(v)$ are the transmission coefficients of the auxiliary problem in the spider domain $\Omega_{v,\varepsilon}$ (i.e. $t_{j,n}^{s,k}(v)$ are the elements of T_v), and $\delta_{j,n}^{s,k} = 1$ if $(s,k) = (j,n)$, $\delta_{j,n}^{s,k} = 0$ if $(s,k) \neq (j,n)$.

Definition 5 *A family of subsets $l(\varepsilon)$ of a bounded closed interval $l \subset R^1$ will be called thin if, for any $\delta > 0$, there exist constants $\beta > 0$ and c_1, independent of δ and ε, and $c_2 = c_2(\delta)$, such that $l(\varepsilon)$ can be covered by c_1 intervals of length δ together with $c_2\varepsilon^{-1}$ intervals of length $c_2e^{-\beta/\varepsilon}$. Note that $|l(\varepsilon)| \to 0$ as $\varepsilon \to 0$.*

Theorem 6 *Let l be a bounded closed interval of the λ-axis which does not contain points $\lambda_{j,n}$. Then there exists $\gamma = \gamma(\omega_j, l) > 0$ and a thin family of sets $l(\varepsilon)$ such that the asymptotic expansion (15) holds on all (finite and infinite) channels $C_{j,\varepsilon}$ uniformly in $\lambda \in l \setminus l(\varepsilon)$ and x in any bounded region of R^d. The function ς in (15) is a vector function on the limiting graph which satisfies equation (16), conditions (17) at infinity, and the GC (18).*

Remarks. 1) It will be shown in the proof of Lemma 11 that for spider domains the estimate of the remainder is uniform for all $x \in R^d$. For general domains, we provide the estimate of the remainder only in bounded regions of R^d in order not to complicate the exposition.

2) The arguments, used to justify the asymptotic behavior of the scattering solutions and prove Theorem 6, can be applied to the study the asymptotic behavior of the outgoing solutions of the non-homogeneous problem (5) as $\varepsilon \to 0$, $\lambda > \lambda_0$. The asymptotics will be expressed in terms of solutions of the corresponding non-homogeneous equation on the limiting graph. One can easily show that the GC can not be chosen independently of f even if we consider only functions f with compact support. However, if the support of f is separated from the junctions then the solution of the non-homogeneous equation on the limiting graph satisfies the same universal GC (18) that appear when scattering solutions are studied. The latter is related to the following fact: the outgoing solution in a narrow channel behaves as a combination of plane waves plus a term which decays exponentially outside of the support of f when $\varepsilon \to 0$.

Note that the GC for the function ς on the limiting graph depend on λ. In fact, there exists an effective matrix potential on Γ which is independent of λ, and allows one to single out the scattering solutions ς on Γ with the same scattering data as for the original problem in Ω_ε. These results will be published elsewhere.

The convergence of the spectrum of the problem in Ω_ε to the spectrum of a problem on the limiting graph has been extensively discussed in physical and mathematical literature (e.g., [4]-[7], [9, 12, 13, 16, 18] and references therein). What makes our paper different is the following: all the publications that we are aware of, are devoted to the convergence of the spectra (or resolvents) only in a small (in fact, shrinking with $\varepsilon \to 0$) neighborhood of λ_0 (bottom of the absolutely continuous spectrum), or below λ_0. Usually, the Neumann BC on $\partial\Omega_\varepsilon$ is assumed. We deal with asymptotic behavior of solutions of the scattering problem in Ω_ε when λ is close to $\widehat{\lambda} > \lambda_0$, and the BC on $\partial\Omega_\varepsilon$ can be arbitrary.

In particular, papers [5], [12], [13], [18] contain the gluing conditions and the justification of the limiting procedure $\varepsilon \to 0$ near the bottom of the spectrum λ_0 under assumption that the Neumann BC is imposed at the boundary of Ω_ε. Note that $\lambda_0 = 0$ for the Neumann BC. Typically, the GC in this case are: the continuity of $\varsigma(s)$ at each vertex v and $\sum_{j=1}^{d} \varsigma_j'(v) = 0$, i.e. the continuity of both the field and the flow. These GC are called Kirchhoff's GC. In the case when the shrinkage rate of the volume of the junction neighborhoods is lower than the one of the area of the cross-sections of the guides, more complex energy dependent or decoupling condition can arise (see [9], [13], [7] for details). Let us stress again that this is the situation near the bottom $\lambda_0 = 0$ of the absolutely continuous spectrum. As follows from Theorem 3, the GC and the small ε asymptotics are different when $\widehat{\lambda} > \lambda_0$.

Both assumptions ($\lambda \to \lambda_0$, and the fact that the BC is the Neumann condition) in the papers above are very essential. The Dirichlet Laplacian near the bottom of the absolutely continuous spectrum $\lambda_0 > 0$ was studied in a recent paper [16] under the condition that the junctions are more narrow than the tubes. It is assumed there that the domain Ω_ε is bounded. Therefore, the spectrum of the operator (1) is discrete. It is proved that the eigenvalues of the operator (1) in $O(\varepsilon^2)$-neighborhood of λ_0 behave asymptotically, when $\varepsilon \to 0$, as eigenvalues of the problem in the disconnected domain that one gets by omitting the junctions, separating the channels in Ω_ε, and adding the Dirichlet conditions on the bottoms of the channels. This result indicates that the waves do not propagate through the narrow junctions when λ is close to the bottom of the absolutely continuous spectrum. A similar result was obtained in [3] for the Schrödinger operator with a potential having a deep strict minimum on the graph, when the width of the walls shrinks to zero.

We also studied the Dirichlet problem for general domains Ω_ε without special assumptions on the geometry of the junctions when, simultaneously, $\varepsilon \to 0$, $\lambda \to \lambda_0$, and the diameters of the guides and junctions have the same order $O(\varepsilon)$. Our conclusion is that, generically, waves do not propagate through the junctions when the frequency is close to the bottom of the absolutely continuous spectrum. Let us stress that this is true both in the case when the diameters of the junctions are smaller than the diameters of the guides, and in the case when they are larger. Some special conditions must be satisfied for waves to propagate if $\lambda \to \lambda_0$. An infinite cylinder, which can be considered as two half-infinite tubes with the junction of the same shape, can be considered as an example of a domain where the propagation of waves at $\lambda = \lambda_0$ is not suppressed. Less trivial examples will be given in our next paper. We do not deal with the problem near the bottom of the absolutely continuous spectrum in this publication. A detailed analysis of this problem will be published elsewhere. However, we show here that the GC on the limiting graph with $\lambda > \lambda_0$, generically, have a limit as $\varepsilon \to 0$, $\lambda \to \lambda_0$, and the limiting conditions are the Dirichlet conditions. To be more exact, the following statement will be proved.

Theorem 7 *1) Assume that the resolvent (12) does not have a pole at $k = \sqrt{\lambda_0}$. Then the scattering matrix (10), defined for $\lambda > \lambda_0$, admits an analytic in $z = \sqrt{\lambda - \lambda_0}$ extension to a neighborhood of the point $z = 0$ and is equal to $-I$ at $z = 0$, where I is the $(m_0 \times m_0)$-identity matrix and m_0 is the*

number of infinite channels $C_{j,\varepsilon}$ with $\lambda_{j,0} = \lambda_0$.

2) Assume that the resolvent of the auxiliary problem in the spider type domain $\Omega_{\nu,\varepsilon}$ does not have a pole at $k = \sqrt{\lambda_0}$. Then the GC (18) have limit as $\lambda \to \lambda_0$ of the form

$$\varepsilon T'_v \frac{d}{dt}\varsigma^{(v)}(t) + 2i\varsigma^{(v)}(t) = 0, \qquad t = 0,$$

where $T'_v = \frac{d}{dz}T_v$. The GC also have limit when $\varepsilon \to 0$, $\lambda \to \lambda_0$ independently. This limit is the Dirichlet condition $\varsigma^{(v)}(0) = 0$.

A simple version of the results presented in this paper (for models admitting the separation of variables) was published in our paper [14].

The next section contains the proofs of Theorem 4 and 3. The statements of these theorems mostly concern problems with a fixed value of ε. Without loss of the generality, one can assume that $\varepsilon = 1$ there. The last section is devoted to the proof of Theorem 6 on asymptotic behavior of the scattering solutions as $\varepsilon \to 0$. Here the dependence of all objects on ε is essential. At the end of the last section, one can find a proof and a short discussion of Theorem 7.

2 Analytic properties of the resolvent R_λ.

We denote by $\Omega_\varepsilon^{(a)}$ the following bounded part of Ω_ε :

$$\Omega_\varepsilon^{(a)} = \Omega_\varepsilon \backslash \bigcup_{j \le m} (C_{j,\varepsilon} \cap \{t > a\}). \tag{19}$$

The next lemma will be needed later.

Lemma 8 *If the homogeneous problem (5), (6) with a real $\lambda > 0$ has a non-trivial solution u, then either $\sqrt{\lambda}$ is an eigenvalue of $-\varepsilon^2\Delta$ and u decays exponentially at infinity, or $\lambda \in \{\lambda_{j,n}\}$ and (13) holds.*

Proof. From the Green formula for u and $\overline{u}$ in the domain $\Omega_\varepsilon^{(a)}$, $a > 0$, it follows that

$$\mathrm{Im} \int_{\partial\Omega_\varepsilon^{(a)}} \frac{\partial u}{\partial \nu}\overline{u} dS = 0,$$

where ν is the unit normal to $\partial\Omega_\varepsilon^{(a)}$ and dS is an element of the surface area. Using the boundary condition (5) we arrive at

$$\mathrm{Im} \int_{\partial\Omega_\varepsilon^{(a)} \backslash \partial\Omega_\varepsilon} u_t \overline{u} dy = 0. \tag{20}$$

This, (6), and the orthogonality of the functions $\varphi_{j,n}$ imply, for $a \to \infty$,

$$\sum_{j,n:\lambda_{j,n} < \lambda} \sqrt{\lambda - \lambda_{j,n}} |a_{j,n}|^2 + O(e^{-\gamma a}) = 0,$$

which justifies the lemma after taking the limit as $a \to \infty$. This completes the proof.

Let $C'_{j,\varepsilon}$ be the channel $C_{j,\varepsilon}$ extended along the whole t axis,

$$C'_{j,\varepsilon} = \{(t, \varepsilon y) : t \in R, \ y \in \omega_j \subset R^{n-1}\}$$

We denote by $R_\lambda^{(j)}$ the resolvent (11) of the operator $-\varepsilon^2\Delta$ in the extended channel $C'_{j,\varepsilon}$. Let $L_a^2(C'_{j,\varepsilon})$ be the set of functions from $L^2(C'_{j,\varepsilon})$ with the support in the region $|t| \le a$, and let $H^2(C_{j,\varepsilon}^b)$ be the Sobolev space of functions in the domain $C'_{j,\varepsilon} \cap \{b < |t| < b+1\}$. Consider the operator

$$R_\lambda^{(j)} : L_{com}^2(C'_{j,\varepsilon}) \to L_{loc}^2(C'_{j,\varepsilon}). \tag{21}$$

The following lemma can be easily proved using the method of separation of variables.

Lemma 9 *(1) The operator (21) admits an analytic continuation from the upper half plane $Im\lambda > 0$ onto the real axis with the branch points at $\lambda = \lambda_{j,n}$, $n = 0, 1, \dots$.*

(2) If $\lambda_{j,m_j} < \lambda < \lambda_{j,m_j+1}$ and $h \in L^2_{com}(C'_j)$ then $R^{(j)}_\lambda h$ has the following behavior as $t \to \pm\infty$

$$R^{(j)}_\lambda h = \sum_{n=1}^{m_j} c^\pm_{j,n} e^{i\frac{\sqrt{\lambda-\lambda_{j,n}}}{\varepsilon}|t|} \varphi_{j,n}(y/\varepsilon) + O(e^{-\gamma(\varepsilon)|t|}), \quad \gamma > 0, \tag{22}$$

where

$$c^\pm_{j,n} = c^\pm_{j,n}(h) = \frac{\varepsilon^{-d}}{2i\sqrt{\lambda - \lambda_{j,n}}} \int_{\omega_{j,\varepsilon}} \int_{-\infty}^{\infty} e^{\mp i\frac{\sqrt{\lambda-\lambda_{j,n}}}{\varepsilon}\tau} \varphi_{j,n}(y/\varepsilon) h(\tau, y) d\tau dy. \tag{23}$$

(3) Let $\lambda \in l$, where l is a bounded closed interval of the real axis such that $\lambda_{j,m_j} < \lambda < \lambda_{j,m_j+1}$ for all $\lambda \in l$. Let $h \in L^2_{3\varepsilon}(C'_{j,\varepsilon})$ and $b \geq 0$. Then there exist positive constants $c = c(l)$ and $\gamma = \gamma(l)$ which are independent of $\lambda \in l$, ε and h, and such that the remainder term r in the right-hand side of (22) has the estimate

$$||r||_{H^2(C^b_{j,\varepsilon})} \leq ce^{-\gamma b/\varepsilon} ||h||_{L^2_{3\varepsilon}(C'_{j,\varepsilon})}.$$

Proof of Theorem 4. The statements of the theorem mostly concern the problem with a fixed value of ε. Without loss of generality, we can assume that $\varepsilon = 1$, and we omit ε in the notations of all objects (Ω_ε, $C_{j,\varepsilon}$, and so on). The dependence on ε will be restored in some parts of the proof, when this dependence on ε is essential.

Step 1. Construction of the resolvent. Let us introduce the following partition of unity on Ω

$$\sum_{j=0}^{m} \phi_j = 1. \tag{24}$$

We fix arbitrary functions $\phi_j \in C^\infty(\Omega)$, $1 \leq j \leq m$, such that $\phi_j = 1$ in the (infinite) channel C_j for $t \geq 2$, $\phi_j = 0$ in C_j for $t \leq 1$ and outside of C_j. The function ϕ_0 is defined as follows $\phi_0 = 1 - \sum_{j \leq m} \phi_j$. We also need functions ψ_j that are equal to one on the supports of φ_j, which will allow us to smoothly extend functions defined only on infinite channels or only in a bounded part of Ω onto the whole domain Ω. We fix functions $\psi_j \in C^\infty(\Omega)$, $1 \leq j \leq m$, such that $\psi_j = 1$ in the infinite channel C_j for $t \geq 1$ (i. e. on the support of ϕ_j), $\psi_j = 0$ outside of C_j. Let $\psi_0 \in C^\infty(\Omega)$ be a function such that $\psi_0 = 1$ on the support of ϕ_0, and $\psi_0 = 0$ in all infinite channels C_j when $t \geq 3$. Note that

$$\psi_j \phi_j = \phi_j, \quad 0 \leq j \leq m. \tag{25}$$

We construct the parametrix (almost resolvent) for the problem (5) in the form

$$P_\lambda : L^2(\Omega) \to L^2(\Omega), \qquad P_\lambda f = \psi_0 R_{\lambda'}(\phi_0 f) + \sum_{j=1}^{m} \psi_j R^{(j)}_\lambda (\phi_j f). \tag{26}$$

where $R_{\lambda'}$ is the resolvent (11) of the operator in Ω with a fixed $\lambda' = i\sigma$, $\sigma > 0$, which will be chosen later, and $R^{(j)}_\lambda$ are resolvents of the negative Dirichlet Laplacians in C_j. If $f \in L^2(\Omega)$ then $\phi_j f = 0$ outside C_j, and we consider $\phi_j f$ as an element of $L^2(C_j)$. Then the operator $R^{(j)}_\lambda$ can be applied to $\phi_j f$ and $R^{(j)}_\lambda(\phi_j f) \in L^2(C_j)$. Since $\psi_j = 0$ at the bottom of C_j and outside of C_j, we consider $\psi_j R^{(j)}_\lambda(\phi_j f)$ as an element of $L^2(\Omega)$ that is equal to zero outside of C_j. In this way, the operator P_λ is well defined for $\lambda \notin [0, \infty)$.

Let us look for a solution $u \in L^2(\Omega)$ of the problem (5) with $\lambda \notin [0, \infty)$ in the form of $u = P_\lambda h$ with unknown $h \in L^2(\Omega)$. Obviously, u satisfies the Dirichlet boundary condition since each term in (26), applied to any h, satisfies the Dirichlet boundary condition. The substitution of $P_\lambda h$ for u in equation (5) with $\lambda \notin [0, \infty)$ (and $\varepsilon = 1$) leads to

$$(-\Delta - \lambda) P_\lambda h = -(\Delta \psi_0)[R_{\lambda'}(\phi_0 h)] - 2\nabla\psi_0 \cdot \nabla[R_{\lambda'}(\phi_0 h)]$$

$$-\psi_0(\Delta+\lambda')[R_{\lambda'}(\phi_0 h)] - (\lambda-\lambda')\psi_0[R_{\lambda'}(\phi_0 h)]$$
$$-\sum_{j=1}^{m}\Big\{(\Delta\psi_j)[R_\lambda^{(j)}(\phi_j f)] + 2\nabla\psi_j\cdot\nabla R_\lambda^{(j)}(\phi_j h) + \psi_j(\Delta+\lambda)[R_\lambda^{(j)}(\phi_j h)]\Big\} = f,$$

Using (25), (24), the last relation can be rewritten in the form

$$h + F_\lambda h = f, \tag{27}$$

where

$$\begin{aligned} F_\lambda h \quad &= \quad -[(\Delta+\lambda-\lambda')\psi_0][R_{\lambda'}(\phi_0 h)] - 2\nabla\psi_0\cdot\nabla[R_{\lambda'}(\phi_0 h)] \\ &\quad -\sum_{j=1}^{m}\Big\{(\Delta\psi_j)[R_\lambda^{(j)}(\phi_j h)] + 2\nabla\psi_j\cdot\nabla R_\lambda^{(j)}(\phi_j h)\Big\}. \end{aligned} \tag{28}$$

Let us show that the operator

$$F_\lambda : L^2(\Omega)\to L^2(\Omega), \qquad \lambda\notin[\lambda_0,\infty), \tag{29}$$

is compact and depends analytically on λ. Indeed, the resolvents $R_{\lambda'}$ and $R_\lambda^{(j)}$ map any function $f\in L^2$ into the solution of the problem (5) in the domains Ω, C_j, respectively. Thus, these operators are bounded as operators from L^2 into the Sobolev spaces H^2. Since the formula (28) contains at most first derivatives of the resolvents, the operator $F_\lambda,\ \lambda\notin[\lambda_0,\infty)$, is bounded if it is considered as an operator from $L^2(\Omega)$ into the Sobolev space $H^1(\Omega)$. Since $\nabla\psi_0=\nabla\psi_j=0$ at points $x\in C_j$ with $t>3$, from (28) it follows that, for any infinite channel C_j,

$$F_\lambda h = 0, \quad x\in C_j\cap\{t>3\}. \tag{30}$$

Hence, the Sobolev imbedding theorem implies that the operator (29) is compact. The analyticity of the operator (29) is obvious since the operators $R_\lambda^{(j)}$ depend analytically on λ, and $R_{\lambda'}$ does not depend on λ.

Now we put $\lambda=\lambda'=i\sigma$ and show that $||F_{i\sigma}||\to 0$ as $\sigma\to\infty$. In fact, since the norm of the resolvent does not exceed the inverse distance from the spectrum, we have that

$$||R_{\lambda'}||,\ ||R_{\lambda'}^{(j)}|| \le 1/\sigma, \tag{31}$$

where the first norm is considered in the space $L^2(\Omega)$ and the second one is in the space $L^2(C_j)$. Multiplying the equation (5), considered in the domain Ω or C_j, by u and integrating over the domain, we get the following relation for the functions $u=R_{\lambda'}f$ and $u=R_{\lambda'}^{(j)}f$, respectively:

$$||\nabla u||_{L^2}^2 - i\sigma\ ||u||_{L^2}^2 = \int uf dx,$$

which implies that

$$||\nabla u||_{L^2}^2 \le |\int uf dx| \le ||u||_{L^2}||f||_{L^2}.$$

Thus,

$$||R_{\lambda'}f||_{H^1(\Omega)},\ ||R_{\lambda'}^{(j)}f||_{H^1(C_j)} \le C\sigma^{-1/2}||f||_{L^2}. \tag{32}$$

Since the formula (28) contains at most first derivatives of the resolvents, estimates (31), (32) imply that $||F_{i\sigma}||\to 0$ as $\sigma\to\infty$.

We fix $\lambda'=i\sigma$ in (26) in such a way that $||F_{\lambda'}||<1$. Then from the analytic Fredholm theorem it follows that the operator

$$(E+F_\lambda)^{-1} : L^2(\Omega)\to L^2(\Omega), \quad \lambda\notin[\lambda_0,\infty), \tag{33}$$

exists and depends meromorphically on λ. From here, (26) and (27) the following representation for the resolvent follows

$$R_\lambda = P_\lambda(E + F_\lambda)^{-1}, \quad \lambda \notin [\lambda_0, \infty). \tag{34}$$

Step 2. Analytic continuation of the resolvent. In order to extend the operator (12) meromorphically into the lower half plane Im$k < 0$ we need to repeat the arguments used to justify (34). Consider the space $L^2_a(\Omega)$ of functions $f \in L^2(\Omega)$ with supports in $\Omega^{(a)}$ (see (19)), i.e. $f = 0$ in the infinite channels C_j when $t > a$. Let $f \in L^2_a(\Omega)$. Without loss of generality, one can assume that $a > 3$. Then (27) and (30) imply that h is also supported in $\Omega^{(a)}$, i.e. F_λ can be considered as an operator in $L^2_a(\Omega)$:

$$F_\lambda : L^2_a(\Omega) \to L^2_a(\Omega), \quad \lambda \notin [0, \infty).$$

Let $\chi = \chi_a(t)$ be a function equal to one when $t \leq a$ and zero when $t > a$. From Lemma 9 it follows that the operators

$$\chi R^{(j)}_{k^2} : L^2_a(C_j) \to L^2_a(C_j), \quad \mathrm{Im}k > 0,$$

admit an analytic continuation into the lower half plane with the branch points at $k = \pm\sqrt{\lambda_{j,n}}$. Further, $u = R^{(j)}_{k^2} f$ satisfies equation (5) with $\lambda = k^2$ for all complex $k \in \mathbf{C}$, and therefore the operators

$$\chi R^{(j)}_{k^2}, \ \chi\nabla R^{(j)}_{k^2} : L^2_a(C_j) \to L^2_a(C_j), \quad k \in \mathbf{C},$$

are compact and analytic in the complex plane $\mathbf{C}$. Since $\chi = 1$ on the supports of $\nabla\psi_j$, $0 \leq j \leq m$, we can insert the factor χ on the left of all the resolvents $R^{(j)}_\lambda$ in (28). From here it follows that the operator

$$F_{k^2} : L^2_a(\Omega) \to L^2_a(\Omega), \quad k \in \mathbf{C},$$

is compact and analytic with branch points at $k = \pm\sqrt{\lambda_{j,n}}$. Hence, the operator

$$(E + F_{k^2})^{-1} : L^2_a(\Omega) \to L^2_a(\Omega), \quad k \in \mathbf{C}, \tag{35}$$

is meromorphic with the branch points at $k = \pm\sqrt{\lambda_{j,n}}$. Together with (26), (34) and the analyticity of the operators $R^{(j)}_{k^2} : L^2_a(C_j) \to L^2_{loc}(C_j)$, $k \in \mathbf{C}$, this implies that the operator (12) admits a meromorphic continuation to the lower half plane with the branch points at $k = \pm\sqrt{\lambda_{j,n}}$ and poles determined by the poles of the operator (35). Obviously, the poles of the operator (35) may have a limiting point only at $\lambda = \infty$.

Step 3. Spectral analysis. First of all note that the existence of the meromorphic extension of the operator (12) together with the Stone formula immediately imply that the operator $H = -\Delta$ does not have singular spectrum. The proof of this fact can be found in [17] (see Theorem XIII.20).

In order to prove the part of statement (1) of the theorem concerning the absolutely continuous spectrum of the operator $H = -\Delta$, we split the domain Ω into pieces by introducing cuts along the bases $t = 0$ of all infinite channels. We denote the new (not connected) domain by Ω', and denote the negative Dirichlet Laplacian in Ω' by H', i. e. H' is obtained from H by introducing additional Dirichlet boundary conditions on the cuts. Obviously, the operator H has the absolutely continuous spectrum described in statement (1) of the theorem. Thus, it remains to show that the wave operators for the couple H, H' exist and complete. The justification of the existence and completeness of the wave operators can be found in [1]. Another option is to derive the latter fact independently using Birman theorem stating that the validity of the inclusion

$$(H - \lambda)^{-n} - (H' - \lambda)^{-n} \in J_1 \tag{36}$$

for some λ and $n \geq 1$ implies the existence and completeness of the wave operators. Here J_1 is the space of operators of the trace class. The inclusion (36) can be derived from (34) and a similar formula for the resolvent of the operator H'. This completes the proof of the statement about the absolutely continuous spectrum.

The discreteness of the set $\{\lambda^{j,\varepsilon}\}$ of eigenvalues follows from the fact that the operator (12) is meromorphic in λ and has poles at $\{\lambda^{j,\varepsilon}\}$. The existence of the poles at $\{\lambda^{j,\varepsilon}\}$ can be derived from the Stone formula. Another proof will be given below.

Let us prove the part of statement (1) concerning the spider domains. If Ω_ε is a spider domain, then there exists a point $\widehat{x}(\varepsilon)$ and an ε-independent domain Ω such that the transformation (see (2))

$$L_\varepsilon : x \to \widehat{x}(\varepsilon) + \varepsilon x, \tag{37}$$

maps Ω into Ω_ε. In order to stress the fact that the operator $H = -\varepsilon^2\Delta$ in the domain Ω_ε depends on ε, we shall denote it by $H^{(\varepsilon)}$. The operator $-\Delta$ in the domain Ω shall be denoted by $H^{(1)}$. Obviously,

$$H^{(\varepsilon)} = L_\varepsilon H^{(1)} L_\varepsilon^{-1}, \tag{38}$$

and this implies the independence of the eigenvalues of the operator $H^{(\varepsilon)}$ of ε. This completes the proof of statement (1).

Step 4, real poles of the resolvent. The first part of statement (2) about the existence of the analytic extension of the resolvent was justified in step 2 of the proof. Now we are going to prove the second part of that statement concerning the set of real poles of the operator (12). We denote this set of poles by K. Let us assume that either u is an eigenfunction of the operator $H = -\Delta$ with an eigenvalue $\lambda = \lambda' > 0$ or u is a non-trivial solution of the homogeneous problem (5), (13) with $\lambda = \lambda' > 0$ (recall that we assume that $\varepsilon = 1$). We are going to show that $k = \pm\sqrt{\lambda'} \in K$. Consider the restrictions u_j of u to the cylinders C_j, $1 \le j \le m$. Let $v_j \in L^2(C_j)$ be the solution of the problem

$$(-\Delta - \lambda)v_j = 0, \qquad x \in C_j; \qquad v_j = 0 \ \text{ on } \partial' C_j, \quad v_j = u_j \ \text{ when } t = 0,$$

where $\lambda \notin [0, \infty)$, and $\partial' C_j$ is the lateral boundary of C_j. The solution $v_j \in L^2(C_j)$ of this problem is unique and can be found by separation of variables. The function u_j satisfies the same equation with the fixed $\lambda = \lambda'$ and the same boundary conditions. It is also defined uniquely by its values at $t = 0$ and can be found by separation of variables. This implies that v_j converges to u as $\lambda \to \lambda' + i0$. Since u is a solution of a homogeneous elliptic problem, $u \in C^\infty$. Thus, u_j is infinitely smooth when $t = 0$, and the convergence $v_j \to u_j$ takes place, for example, in the Sobolev space H^2 on the part of the cylinder C_j where $0 \le t \le 2$. Let

$$v = \sum_{j=1}^{m} \phi_j v_j + \phi_0 u \in L^2(C_j), \quad \lambda \notin [0, \infty),$$

where $\{\phi_j\}$ is the partition of unity which was introduced above. The function u can not be equal to zero identically on $\Omega \backslash \cup C_j$ due to the uniqueness of the solution of the Cauchy problem for the operator $-\Delta - \lambda'$. Thus

$$||v||_{L^2(\Omega\backslash\cup C_j)} = ||u||_{L^2(\Omega\backslash\cup C_j)} = c_0 > 0. \tag{39}$$

On the other hand,

$$(-\Delta - \lambda)v = -\sum_{j=1}^{m} [(\Delta\phi_j)v_j + 2\nabla\phi_j \cdot \nabla v_j] - (\lambda - \lambda')\phi_0 u - (\Delta\phi_0)u - 2\nabla\phi_0 \cdot \nabla u.$$

Thus, $(-\Delta - \lambda)v \in L_a^2(\Omega)$. From the convergence $v_j \to u$ and (24) it follows that $(-\Delta - \lambda)v$ tends to zero in $L_a^2(\Omega)$ as $\lambda \to \lambda' + i0$. Together with (39) this provides the existence of the pole of the operator (12) at $k = \sqrt{\lambda'}$. The pole at $k = -\sqrt{\lambda'}$ exists due to the relation $R_\lambda = \overline{R_{\overline{\lambda}}}$. Hence, $\pm\sqrt{\lambda'} \in K$.

Now let us assume that at least one of the points $\pm\sqrt{\lambda'}$ belongs to K. The relation $R_\lambda = \overline{R_{\overline{\lambda}}}$ implies that the second point also belongs to K, i.e. $\sqrt{\lambda'} \in K$, and there exist $a > 0$ and $f \in L_a^2(\Omega)$ such that

$$w; = R_\lambda f = \frac{u(x)}{(\lambda-\lambda')^n} + \frac{v(x,\lambda)}{(\lambda-\lambda')^{n-1}}; \quad ||v||_{L^2(\Omega^{(a+2)})} \leq c, \quad \lambda \to \lambda' + i0, \tag{40}$$

where $n \geq 1$ and u does not vanish identically. In fact, n can not exceed one, but it is not important for us now. Obviously,

$$(-\Delta - \lambda')u = 0, \quad x \in \Omega; \quad u = 0 \ \text{ on } \partial\Omega. \tag{41}$$

From here and Lemma 8 it follows that in order to complete the proof of the second statement of the theorem it is sufficient to show that the asymptotic expansion (6) holds for the function u.

Note that (41) implies that $u \in C^\infty$. Since $f = 0$ in all infinite channels C_j when $t > a$, from relation (40) it follows that

$$(-\Delta - \lambda)v = (\lambda - \lambda')u, \quad x \in C_j \cap \{t > a\}; \quad v = 0 \quad \text{on } \partial\Omega.$$

From here, the estimate in (40), and standard local a priory estimates for solutions of elliptic problems it follows that for any vector α

$$|\frac{\partial^\alpha v}{\partial x^\alpha}| \leq c(\alpha), \quad x \in C_j \cap \{a + \frac{3}{2} > t > a + \frac{1}{2}\}, \quad \lambda \to \lambda' + i0,$$

and therefore

$$\frac{\partial^\alpha[(\lambda-\lambda')^n w]}{\partial x^\alpha} \to \frac{\partial^\alpha u}{\partial x^\alpha} \tag{42}$$

uniformly on $C_j \cap \{t = a+1\}$ as $\lambda \to \lambda' + i0$. We restrict the functions $(\lambda - \lambda')^n w$ and u to $C_j \cap \{t = a+1\}$ and expand the restrictions with respect to the basis $\{\varphi_{j,n}\}$ of the operator $-\Delta$ in the cross section of the channel C_j. Let $\gamma_{j,n}(\lambda)$ and $\gamma_{j,n}^0$ be the coeficients of these expansions. Then (42) implies that for any β,

$$|\gamma_{j,n}(\lambda) - \gamma_{j,n}^0| < c_\beta n^{-\beta}, \quad \lambda \to \lambda' + i0. \tag{43}$$

The function $\widehat{w} := (\lambda - \lambda')^n w$ satisfies the following relations in $C_j \cap \{t \geq a+1\}$:

$$(-\Delta - \lambda)\widehat{w} = 0, \quad \widehat{w} = 0 \ \text{ for } \ x \in \partial C_j \cap \{t > a+1\}, \quad \widehat{w}|_{t=a+1} = \sum_n \gamma_{j,n}(\lambda)\varphi_{j,n}(y),$$

where $\lambda \notin [0,\infty)$. One can find the solution $\widehat{w} \in L^2$ of this problem by the method of separation of variables and then pass to the limit as $\lambda \to \lambda' + i0$ using (43). This leads to the asymptotic expansion (6) for u and completes the proof of the second statement of Theorem 4.

Step 5, the proof of the last two statements of the theorem. If $k = \sqrt{\lambda'}$, $\lambda' > 0$, is not a pole or a branch point of R_{k^2} then

$$w := R_\lambda f = u(x) + (\lambda - \lambda')v(x,\lambda); \quad ||v||_{L^2(\Omega^{(a)})} \leq c(a), \quad \lambda \to \lambda' + i0,$$

where $a > 0$ is arbitrary and

$$(-\Delta - \lambda')u = f, \quad x \in \Omega; \quad u = 0 \ \text{ on } \partial\Omega.$$

In order to prove the third statement of the theorem, we need only to show that the asymptotic expansion (6) holds for u. It can be done exactly in the same way as it was done for function u in (40) by representing u in $C_j \cap \{t > a+1\}$ as the limit of functions w as $\lambda \to \lambda' + i0$.

In order to prove the last statement of the theorem one can look for the solution $\Psi = \Psi_{s,k}$ of the scattering problem in the form

$$\Psi = \phi_s e^{-i\sqrt{\lambda-\lambda_{s,k}}t}\varphi_{s,k}(y) + u,$$

where ϕ_s is the function from the partition of unity (24). This reduces problem (7), (8) to the uniquely solvable problem (5), (6) for u.

This completes the proof of Theorem 4.

Proposition 10 *Let $\Psi_{s,k}$ and $\Psi_{s',k'}$ be two scattering solutions, and let $a_{j,n}^{s,k}$ be the transmission coefficients for the scattering solution $\Psi_{s,k}$. Then*

1) The following energy conservation law is valid:

$$\sum_{j,n}\sqrt{\lambda-\lambda_{j,n}}|a_{j,n}^{s,k}|^2 = \sum_{j=1}^{m}\sum_{n=0}^{m_j}\sqrt{\lambda-\lambda_{j,n}}|a_{j,n}^{s,k}|^2 = \sqrt{\lambda-\lambda_{s,k}}.$$

2) If these solutions correspond to different incident waves ($(s,k)\neq(s',k')$), then

$$\sum_{j,n}\sqrt{\lambda-\lambda_{j,n}}a_{j,n}^{s,k}\overline{a}_{j,n}^{s',k'} = 0.$$

Proof. Since the statement concerns the problem with a fixed value of ε, one can put $\varepsilon = 1$ and omit ε in the notations Ω_ε, $\Omega_\varepsilon^{(a)}$. Green's formula for $\Psi_{s,k}$ and $\overline{\Psi}_{s',k'}$ in the domain $\Omega^{(a)}$ implies, similarly to (20), that

$$\int_{\partial\Omega^{(a)}\setminus\partial\Omega}[(\Psi_{s,k})_t\overline{\Psi}_{s',k'} - \Psi_{s,k}(\overline{\Psi}_{s',k'})_t]dy = 0.$$

From here, (8), and the orthogonality of the functions $\varphi_{j,n}$ it follows that

$$\sum_{j,n}\sqrt{\lambda-\lambda_{j,n}}a_{j,n}^{s,k}\overline{a}_{j,n}^{s',k'} - \sqrt{\lambda-\lambda_{s,k}}\overline{a}_{j,n}^{s,k}e^{-2i\sqrt{\lambda-\lambda_{s,k}}a}$$

$$+\sqrt{\lambda-\lambda_{s',k'}}a_{j,n}^{s',k'}e^{2i\sqrt{\lambda-\lambda_{s',k'}}a} - \sqrt{\lambda-\lambda_{s,k}}\delta + O(e^{-\gamma a}) = 0, \qquad a\to\infty,$$

where $\delta = 1$ if $\begin{pmatrix} s \\ k \end{pmatrix} = \begin{pmatrix} s' \\ k' \end{pmatrix}$, and $\delta = 0$ otherwise. We take the average with respect to $a \in (A, 2A)$ and pass to the limit as $A\to\infty$. Then we get

$$\sum_{j,n}\sqrt{\lambda-\lambda_{j,n}}a_{j,n}^{s,k}\overline{a}_{j,n}^{s',k'} = \sqrt{\lambda-\lambda_{s,k}}\delta,$$

which justifies both statements of the proposition. This completes the proof.

Proof of Theorem 3. Proposition 10 is equivalent to the relation $A^*A = I$ for $A = D^{1/2}TD^{-1/2}$, which provides the unitarity of the matrix A. If one applies Green's formula to the scattering solutions $\Psi_{s,k}$ and $\Psi_{s',k'}$, then the arguments used in the proof of Proposition 10 lead to the symmetry of $D^{1/2}TD^{-1/2}$. This completes the proof of Theorem 3.

3 Asymptotic behavior of scattering solutions as $\varepsilon\to 0$.

We start with a study of scattering solutions in spider domains Ω_ε

Lemma 11 *Theorem 6 is valid for spider domains.*

Proof. The transformation L_ε^{-1}, see (37), maps the spider domain Ω_ε into the $\varepsilon-$independent domain Ω with the channels C_j, $1\leq j\leq m$, The coordinates $(\widehat{t},\widehat{y})$ in C_j are related to coordinates (t,y) in $C_{j,\varepsilon}$ via the formulas

$$\widehat{t}=t/\varepsilon,\quad \widehat{y}=y/\varepsilon. \tag{44}$$

The scattering solution $\widehat{\Psi}=\widehat{\Psi}_{s,k}$ of the problem in Ω has the form similar to (8):

$$\begin{aligned}\widehat{\Psi}_{s,k} &= \delta_{s,j}e^{-i\sqrt{\lambda-\lambda_{s,k}}\widehat{t}}\varphi_{s,k}(\widehat{y})+\sum_{n=0}^{m_j}t_{j,n}e^{i\sqrt{\lambda-\lambda_{j,n}}\widehat{t}}\varphi_{j,n}(\widehat{y})+O(e^{-\gamma\widehat{t}}),\\ x &\in C_j,\quad \widehat{t}\to\infty.\end{aligned}$$

Since $\widehat{\Psi}_{s,k}$ is a smooth function, the remainder term $\widehat{r}$ in the formula above can be estimated for all values of $\widehat{t}$:

$$|\widehat{r}|\leq Ce^{-\gamma\widehat{t}},\quad x\in C_j.$$

Since the scattering solutions in the domains Ω_ε and Ω are related via the formula $\Psi_{s,k}(x)=\widehat{\Psi}_{s,k}(L_\varepsilon^{-1}x)$, it follows that

$$\begin{aligned}\Psi_{s,k}^{(\varepsilon)} &= \delta_{s,j}e^{-i\frac{\sqrt{\lambda-\lambda_{s,k}}}{\varepsilon}t}\varphi_{s,k}(y/\varepsilon)+\sum_{n=0}^{m_j}t_{j,n}e^{i\frac{\sqrt{\lambda-\lambda_{j,n}}}{\varepsilon}t}\varphi_{j,n}(y/\varepsilon)+r^{(\varepsilon)},\\ |r^{(\varepsilon)}| &\leq Ce^{-\gamma t/\varepsilon},\quad x\in C_j.\end{aligned} \tag{45}$$

Thus, the asymptotic expansion (15), (17) is valid, and it only remains to show that the GC (18) holds for vectors $\varsigma=\varsigma_{s,k}$ determined by (45) (the definition of these vectors is given in the paragraph above formula (18)). We form the matrix $\Sigma=\Sigma(t)$ with columns $\varsigma_{s,k}$ taking them in the same order as the order chosen for elements in each of these vectors (first we put columns with $s=1$ and $k=1,2,...,m_1$, then columns with $s=2$, and so on). From (45) it follows that

$$\Sigma(0)=I+T,\quad \Sigma'(0)=\frac{i}{\varepsilon}D(-I+T),$$

where T is the scattering matrix, I is the identity matrix of the same size, and D is the diagonal matrix of the same size with elements $\sqrt{\lambda-\lambda_{j,n}}$ on the diagonal. Hence, $\varepsilon(I+T)D^{-1}\Sigma'(0)+i(I-T)\Sigma(0)=0$ and GC (18) holds for the columns of the matrix Σ. This completes the proof of the lemma.

The following two lemmas about spider domains will be needed in order to prove Theorem 6 for general domains. Let R_λ be the resolvent of the operator $H=-\varepsilon^2\Delta$ in a spider domain Ω_ε, and let $\widehat{R}_\lambda$ be the resolvent of the similar operator $H=-\Delta$ in the domain Ω which is the image of Ω_ε under the map L_ε^{-1}, see (37). Note that the operator R_λ and its domain, $L^2(\Omega_\varepsilon)$, depend on ε, while the operator $\widehat{R}_\lambda$ is ε-independent. Formula (38) implies

Lemma 12 *The following relation holds*

$$R_\lambda=L_\varepsilon\widehat{R}_\lambda L_\varepsilon^{-1}.$$

Let us fix m constants $t_j>0$, $1\leq j\leq m$. Let Ω_ε be a spider domain with the channels $C_{j,\varepsilon}$, $1\leq j\leq m$. Consider slices $D_{j,\varepsilon}$ of $C_{j,\varepsilon}$ defined by the inequalities $|t-t_j|\leq 3\varepsilon$. Let Ω_ε' be a bounded domain which is obtained from Ω_ε by cutting off the infinite parts of channels $C_{j,\varepsilon}$ on which $t\geq\frac{3}{4}t_j$. Let a function $h\in L^2(\Omega_\varepsilon)$ be supported in one of the domains $D_{j,\varepsilon}$, for example, with $j=s$.

Below, when the resolvent R_λ of the operator $H=-\varepsilon^2\Delta$ in Ω_ε is considered with λ belonging to the continuous spectrum of the operator, R_λ is understood in the sense of the analytic extension described in Theorem 4. We denote the Sobolev spaces of functions which are square integrable together with their derivatives of up to the second order by $H^2(\Omega_\varepsilon')$ and $H^2(D_{j,\varepsilon})$.

Lemma 13 *Let Ω_ε be a spider domain. Let l be a bounded closed interval of the λ-axis that does not contain points $\lambda_{j,n}$, and let a function $h \in L^2(\Omega_\varepsilon)$ be supported in the domain $D_{s,\varepsilon}$. Then there exists $\gamma = \gamma(\omega_j, l) > 0$ such that*

$$(1) \qquad R_\lambda h = \sum_{k=0}^{m_s} c_{s.k}\Psi_{s,k} + r_0 \quad in \quad \Omega'_\varepsilon, \quad |r_0(x)| \leq \frac{Ce^{-\gamma/\varepsilon}}{\Pi_{\lambda^j \in l}|\lambda - \lambda^j|}||h||_{L^2(\Omega_\varepsilon)}, \tag{46}$$

where $\Psi_{s,k}$ are scattering solutions, the coefficients $c_{s.k} = c^-_{s.k}(h)$ are given by (23), and λ^j are eigenvalues of the operator H in Ω_ε (see statement (1) of Theorem 4);

$$(2) \qquad R_\lambda h = \delta_{s,j}R^{(s)}_\lambda h + \sum_{k=0}^{m_s} c_{s.k}\sum_{n=0}^{m_j} t^{s,k}_{j,n} e^{i\frac{\sqrt{\lambda-\lambda_{j,n}}}{\varepsilon}t}\varphi_{j,n}(y/\varepsilon) + r_j \quad in \quad D_{j,\varepsilon}, \tag{47}$$

$$where \quad ||r_j||_{H^2(D_{j,\varepsilon})} \leq \frac{Ce^{-\gamma/\varepsilon}}{\Pi_{\lambda^j \in l}|\lambda - \lambda^j|}||h||_{L^2(\Omega_\varepsilon)}.$$

Here $\delta_{s,j}$ is the Kronecker symbol ($\delta_{s,j} = 1$ if $s = j$, $\delta_{s,j} = 0$ if $s \neq j$), $R^{(s)}_\lambda$ is the resolvent of $-\Delta$ in the extended channel $C'_{s,\varepsilon}$ (channel $C_{s,\varepsilon}$ extended to $-\infty$ along the t axis), $c_{s.k} = c^-_{s.k}(h)$, and $t^{s,k}_{j,n}$ are the transmission coefficients (see the remark following definition 2).

Proof. Let a function $\alpha \in C^\infty(\Omega_\varepsilon)$ have the form: $\alpha = 1$ in $C_{s,\varepsilon}$ when $t > \frac{7}{8}t_j + \varepsilon$, $\alpha = 0$ in $\Omega_\varepsilon \backslash C_{s,\varepsilon}$, and $\alpha = 0$ in $C_{s,\varepsilon}$ when $t < \frac{7}{8}t_j$. Consider the function

$$u = \alpha R^{(s)}_\lambda h + \sum_{k=0}^{m_s} c_{s.k}[\Psi_{s,k} - \alpha e^{-i\frac{\sqrt{\lambda-\lambda_{s,k}}}{\varepsilon}t}\varphi_{s,k}(y/\varepsilon)]. \tag{48}$$

Obviously, $u = 0$ on $\partial\Omega_\varepsilon$, since each term in the right hand side above satisfies the Dirichlet boundary condition. Furthermore,

$$-\varepsilon^2\Delta u - \lambda u = h - \varepsilon^2[\nabla\alpha \cdot \nabla R^{(s)}_\lambda h + (\Delta\alpha)R^{(s)}_\lambda h - \nabla\alpha \cdot \nabla g - (\Delta\alpha)g], \tag{49}$$

where

$$g = \sum_{k=0}^{m_s} c_{s.k}e^{-i\frac{\sqrt{\lambda-\lambda_{s,k}}}{\varepsilon}t}\varphi_{s,k}(y/\varepsilon).$$

The right hand side in (49) has the form $h + h_1$, where h_1 is supported in the slice $\frac{7}{8}t_s \leq t \leq \frac{7}{8}t_s + \varepsilon$ of $C_{s,\varepsilon}$. From Lemma 9 it follows that

$$||h_1||_{L^2(\Omega_\varepsilon)} \leq Ce^{-\gamma/\varepsilon}||h||_{L^2(\Omega_\varepsilon)}.$$

It is also clear that the behavior of the function u at infinity is described by (6). Hence, $u = R_\lambda(h + h_1)$ due to the statement (3) of Theorem 4. From here and (48) it follows that

$$R_\lambda h = \alpha R^{(s)}_\lambda h + \sum_{k=0}^{m_s} c_{s.k}(\Psi_{s,k} - \alpha e^{-i\frac{\sqrt{\lambda-\lambda_{s,k}}}{\varepsilon}t}\varphi_{s,k}(y/\varepsilon)) - R_\lambda h_1. \tag{50}$$

This implies equality (46) with $r_0 = -R_\lambda h_1$. Let Ω''_ε be obtained from Ω_ε by cutting off the parts of channels $C_{j,\varepsilon}$ where $t \geq \frac{7}{8}t_j$. Since operator (12) is meromorphic (due to Theorem 4) and has poles of at most first order due to the Stone formula, $||r_0||_{L^2(\Omega''_\varepsilon)}$ can be estimated by the right hand side of inequality (46). Since $(-\varepsilon^2\Delta - \lambda)r_0 = 0$ in Ω''_ε and $r_0 = 0$ on the lateral side of $\partial\Omega''_\varepsilon$, standard a priory estimates for elliptic equations lead to the estimates on r_0 in Sobolev norms in Ω'_ε. These estimates, together with Sobolev imbedding theorems, justify the estimate (46). Similarly, (47) follows from (50) and Lemma 12. This completes the proof of the lemma.

We need two more auxiliary statements in order to prove Theorem 6.

Lemma 14 *Let a real-valued function f belong to $C^{n+1}(R^1)$ and*

$$||f||_{C^{n+1}} = \sum_{k=0}^{n+1} \sup_x |f^{(k)}| = A^+ < \infty, \tag{51}$$

$$\sum_{k=0}^{n} |f^{(k)}(x)| \geq A^- > 0, \qquad x \in R^1. \tag{52}$$

Then for any $\sigma \leq A^-/2$, the set $\Gamma_\sigma = \{x : |f(x)| \leq \sigma\}$ has the following structure. There exists a constant c which depends only on $A^\pm$ and n and such that, for any bounded interval $\Delta \subset R^1$,

a) the number of connected components of Γ_σ in Δ is finite and does not exceed $c(|\Delta|+1)$,

b) the measure of each connected component of Γ_σ in Δ does not exceed $c\sigma^{1/n}$.

Remark. The last estimate can not be improved. In fact, if $f(x) = \sin^n x$ then $\Gamma_\sigma \cap [-\frac{\pi}{2}, -\frac{\pi}{2}] \sim 2\sigma^{1/n}$.

Proof. We shall denote by c_j different constants which depend on $A^\pm$ and n but not on f. If $x \in \Gamma_\sigma$ then (52) implies that

$$\sum_{k=1}^{n} |f^{(k)}(x)| \geq A^-/2,$$

and therefore, $|f^{(k)}(x)| \geq A^-/2n$ for the chosen x and some $k = k(x)$, $1 \leq k \leq n$. Since $|f^{(k+1)}| \leq A^+$, $x \in R^1$, there exists an interval Δ_x such that $x \in \Delta_x$, $|f^{(k)}(x)| \geq A^-/4n$ on Δ_x, and $|\Delta_x| = c_0 = \frac{A^-}{4nA^+}$. The set of intervals Δ_x covers $\Gamma_\sigma \cap \Delta$. Hence, one can select a finite number of intervals Δ_x covering $\Gamma_\sigma \cap \Delta$. Then one can omit some of them in such a way that the remaining intervals still cover $\Gamma_\sigma \cap \Delta$ with multiplicity at most two. This leaves us with at most $2(\frac{|\Delta|}{c_0}+1) \leq c_1(|\Delta|+1)$ intervals Δ_x covering $\Gamma_\sigma \cap \Delta$. Thus, it is enough to prove the lemma for an individual interval Δ' (one of the intervals Δ_x) such that $|\Delta'| = c_0$ and $|f^{(k)}(x)| \geq c_2$ on Δ' for some fixed value of k, $1 \leq k \leq n$.

Equations $f(x) = \pm\sigma$ have at most k solutions on Δ'. In fact, if there exist $k+1$ points where $f(x) = \sigma$ then there are k intermediate points where $f'(x) = 0$. Thus, there are $k-1$ points where $f''(x) = 0$, and so on. Finally, there has to be a point where $f^{(k)}(x) = 0$. This contradicts the assumption that $|f^{(k)}(x)| \geq c_2$ on Δ'. Hence, the set $\Gamma_\sigma \cap \Delta'$ consists of at most $k+1$ intervals. It remains only to show that the length of these intervals does not exceed $c\sigma^{1/k}$. In order to estimate this length, we assume that there is an interval $[x_1, x_1+h]$ where $|f(x)| \leq \sigma$, $|f^{(k)}(x)| \geq c_2$. Put $h' = h/k$ and consider the k-th difference

$$\Delta_k = f(x_1) - \begin{pmatrix} k \\ 1 \end{pmatrix} f(x_1+h') + \begin{pmatrix} k \\ 2 \end{pmatrix} f(x_1+2h') - ... + (-1)^k f(x_1+kh'). \tag{53}$$

There exists a point $\xi_k \in [x_1, x_1+h]$ such that $\Delta_k = (h')^k f^{(k)}(\xi_k)$. Thus, $|\Delta_k| \geq \frac{c_2 h^k}{k^k} = c_3 h^k$. On the other hand, from (53) and the estimate $|f(x)| \leq \sigma$ it follows that $|\Delta_k| \leq \sigma 2^k$. Hence, $c_3 h^k \leq \sigma 2^k$, i.e. $h \leq c\sigma^{1/k}$. This completes the proof of the lemma.

Lemma 15 *Let a set of functions $f_\varepsilon = f_\varepsilon(\lambda)$, $\varepsilon \to 0$, on a closed interval $l \subset R^1$, have the form*

$$f_\varepsilon = \sum_{j=1}^{M} C_j(\lambda) e^{i\frac{g_j(\lambda)}{\varepsilon}}, \tag{54}$$

where functions $C_j(\lambda)$ and real valued, functions $g_j(\lambda)$ are analytic, there are no two functions $g_j(\lambda)$ whose difference is a constant, and

$$\sum_{j=1}^{M} |C_j(\lambda)| \geq 1. \tag{55}$$

Then, for any $\eta > 0$, the set

$$\Gamma_\eta(\varepsilon) = \{\lambda : |f_\varepsilon(\lambda)| \le e^{-\eta/\varepsilon}\} \tag{56}$$

is thin (see the definition in the introduction).

Proof. Consider the set Γ_0 where $g_i'(\lambda) = g_j'(\lambda)$ for some $i \ne j$. Due to the analyticity of the functions $g_j(\lambda)$, this set consists of a finite number points. Let us denote the number of points in Γ_0 by c_1. Let Γ^δ be the $\delta/2$-neighborhood of Γ^0. Then

$$|g_i'(\lambda) - g_j'(\lambda)| \ge a(\delta) > 0, \quad i \ne j, \quad \lambda \in l \setminus \Gamma^\delta. \tag{57}$$

Consider the functions

$$\widehat{f_\varepsilon}(\mu) = \sum_{j=1}^{M} C_j(\varepsilon\mu) e^{i\frac{g_j(\varepsilon\mu)}{\varepsilon}}, \qquad \varepsilon\mu \in l \setminus \Gamma^\delta.$$

For any k,

$$\frac{d^k}{d\mu^k}\widehat{f_\varepsilon}(\mu) = \sum_{j=1}^{M} [g_j'(\varepsilon\mu)]^k C_j(\varepsilon\mu) e^{i\frac{g_j(\varepsilon\mu)}{\varepsilon}} + O(\varepsilon), \tag{58}$$

We move the remainders to the left hand side and consider (58) with $1 \le k \le M$ as equations for unknowns $C_j(\varepsilon\mu)e^{i\frac{g_j(\varepsilon\mu)}{\varepsilon}}$. The matrix of this system of equations with the elements $a_{j,k} = [g_j'(\varepsilon\mu)]^k$ is a Vandermond matrix, and its determinant is bounded from below due to (57). This and (55) imply that

$$\sum_{j=1}^{M} |\frac{d^k}{d\mu^k}\widehat{f_\varepsilon}(\mu)| \ge A^-(\delta) > 0$$

if ε is small enough. It also follows from (58) that

$$\sum_{j=1}^{M+1} |\frac{d^k}{d\mu^k}\widehat{f_\varepsilon}(\mu)| \le A^+.$$

Hence, Lemma 14 is applicable to at least one of the functions $\mathrm{Re}\widehat{f_\varepsilon}(\mu)$ or $\mathrm{Im}\widehat{f_\varepsilon}(\mu)$ on each connected interval of the set $l \setminus \Gamma^\delta$ stretched by a factor of ε^{-1}. Since we have at most $c_1 + 1$ those intervals, this implies that the set $\{\lambda : |f_\varepsilon(\lambda)| \le \sigma\}$ can be covered by Γ^δ and $c_2(\delta)\varepsilon^{-1}$ intervals of length $c_2(\delta)\sigma^{1/M}$. We take $\sigma = e^{-\eta/\varepsilon}$, and this completes the proof of the lemma.

Proof of Theorem 6. The proof is based on a representation of the resolvent R_λ of the problem in Ω_ε through the resolvents $R_{v,\lambda}$ of the operators $H = -\varepsilon^2\Delta$ in the spider domains $\Omega_{v,\varepsilon}$, formed by an individual junction, which corresponds to a vertex v, and all the channels with an end at this junction, where the channels are extended to infinity if they have finite length. Let us consider the slices $D_{j,\varepsilon}$ of the finite channels $C_{j,\varepsilon}$, $j > m$, defined by the conditions $t_j \le t \le t_j + 3\varepsilon$ where $t_j = 4l_j/5$. We construct the following partition of the unity on Ω_ε :

$$\sum_{v \in V} \phi_v = 1,$$

where V is the set of all the vertices v of the limiting graph Γ, $\phi_v \in C^\infty(\Omega_\varepsilon)$, and is defined as follows. The function ϕ_v is equal to one on the junction J_v, which corresponds to the vertex v, on the infinite channels adjacent to J_v and on the parts of the finite channels adjacent to J_v where $t \le t_j + \varepsilon$. The function ϕ_v is equal to zero on the parts of finite channels adjacent to J_v where $t \ge t_j + 2\varepsilon$, and also on all the other junctions and channels which are not adjacent to J_v. Let $\psi_v \in C^\infty(\Omega_\varepsilon)$, $\psi_v = 1$ on the support of ϕ_v, $\psi_v = 0$ on the parts of finite channels adjacent to J_v where $t \ge t_j + 3\varepsilon$, and also on all other junctions and channels which are not adjacent to J_v.

We fix a vertex $v = \widehat{v}$. Let $J_{\widehat{v}}$ be the corresponding junction of Ω_ε. We choose the parametrization on Γ in such a way that the value $t = 0$ on all the edges adjacent to $\widehat{v}$ corresponds to $\widehat{v}$. The origin ($t = 0$) on all the other edges can be chosen at any end of the edge. We are going to justify the asymptotic expansion (15) in the domain $C(\widehat{v})$ consisting of the infinite channels adjacent to $J_{\widehat{v}}$ and the parts $t < 3l_j/5$ of the finite channels $C_{j,\varepsilon}$ adjacent to $J_{\widehat{v}}$. Moreover, it will be shown that the function ς in the asymptotic expansion satisfies equation (16), conditions (17) at infinity, and the GC (18). Since $\widehat{v}$ is arbitrary and the union of all domains $C(\widehat{v})$, $\widehat{v} \in V$, covers all the channels, the validity of (15) in $C(\widehat{v})$ justifies the statements of Theorem 6.

Let us show that the asymptotic expansion (15) in $C(\widehat{v})$ for any scattering solution $\Psi^{(\varepsilon)}_{s,k}$ follows from a similar expansion for functions of the form $u = R_\lambda f$, where $f \in L^2(\Omega_\varepsilon)$ is supported in $\cup D_{j,\varepsilon}$. In fact, let $u = \psi_{v_1}\Psi^{(\varepsilon)}_{s,k,v_1}$, where the vertex $v_1 = v_1(s)$ corresponds to the first junction J_{v_1} encountered by the incident wave, $\Psi^{(\varepsilon)}_{s,k,v_1}$ is the solution of the scattering problem in the spider domain $\Omega_{v_1,\varepsilon}$, and the function u is considered as a function in Ω_ε which is equal to zero outside of the support of ψ_{v_1}. Then

$$(-\varepsilon^2\Delta - \lambda)u = f, \quad f := -\varepsilon^2[\nabla\psi_{v_1}\cdot\nabla\Psi^{(\varepsilon)}_{s,k,v_1} + (\Delta\psi_{v_1})\Psi^{(\varepsilon)}_{s,k,v_1}].$$

Obviously, $f \in L^2(\Omega_\varepsilon)$ and f is supported in $\cup D_{j,\varepsilon}$. From statement (3) of Theorem 4 it follows that there exists the unique outgoing solution $v = R_\lambda f$ of the equation $(-\varepsilon^2\Delta - \lambda)v = f$, $\lambda \in l \setminus \{\lambda^j\}$. Then

$$\Psi^{(\varepsilon)}_{s,k} = \psi_{v_1}\Psi^{(\varepsilon)}_{s,k,v_1} - R_\lambda f,$$

since this function satisfies (7) and (8). From here and Lemma 11 it follows that the asymptotic expansion (15) and the properties of ς mentioned in Theorem 6 hold for $\Psi^{(\varepsilon)}_{s,k}$ in $C(\widehat{v})$ if the corresponding properties are valid for $R_\lambda f$ in $C(\widehat{v})$. Hence, the proof of the theorem will be complete as soon as we show that, for any $f \in L^2(\Omega_\varepsilon)$ with the support in $\cup D_{j,\varepsilon}$, the function $u = R_\lambda f$ has expansion (15) in $C(\widehat{v})$ with $\beta_{j,n} = 0$ and ς satisfying the GC (18).

Consider the operator P_λ defined by the formula

$$P_\lambda h = \sum_{v\in V}\psi_v R_{v,\lambda}(\phi_v h), \qquad \lambda \in l, \tag{59}$$

where $h \in L^2(\Omega_\varepsilon)$ is supported in $\cup D_{j,\varepsilon}$, l is defined in the statement of Theorem 6, and the resolvents $R_{v,\lambda}$ for real $\lambda \in l$ are understood in the sense of Theorem 4. We look for $u = R_\lambda f$ in the form of $P_\lambda h$ with an unknown $h \in L^2(\Omega_\varepsilon)$. This leads to the equation (compare with (27), (28))

$$h + F_\lambda h = f, \qquad F_\lambda h = -\varepsilon^2\sum_{v\in V}[2\nabla\psi_v\cdot\nabla R_{v,\lambda}(\phi_v h) + (\Delta\psi_v)R_{v,\lambda}(\phi_v h)]. \tag{60}$$

From here, similarly to (34), it follows that

$$R_\lambda f = P_\lambda(I + F_\lambda)^{-1}f \tag{61}$$

for $\mathrm{Im}\lambda > 0$. Similarly to (35), one can show that the operator

$$(I + F_\lambda)^{-1} : L^2_a(\Omega_\varepsilon) \to L^2_a(\Omega_\varepsilon) \tag{62}$$

admits a meromorphic extension into the lower half plane $\mathrm{Im}\lambda \le 0$ with the branch points at $\lambda = \lambda_{j,n}$. The only difference is that now we use operators $R_{v,\lambda}$ instead of $R^{(j)}_\lambda$, and $R_{v,\lambda}$ depend meromorphically on λ, while $R^{(j)}_\lambda$ are analytic in λ. So, one needs to refer to a version of an analytic Fredholm theorem where the operator may have poles (with residues of finite ranks). This version of the theorem can be found in [2], and applications of this theorem to operators similar to (62) can be found in [19], [20]. Hence, formula (61) is established for all complex λ.

All the operators in (61) depend on ε. The function $(I+F_\lambda)^{-1}f$ is meromorphic in λ, and its poles depend on ε. In order to find a set on the interval l where the operator $(I+F_\lambda)^{-1}$ exists and is bounded uniformly in λ we shall use the following reduction of equation (60) to a system of equations where the domains of the operators do not depend on ε.

Recall that f is supported in $\cup D_{j,\varepsilon}$. Formula (60) for F_λ implies that the function $F_\lambda h$ is also supported in $\cup D_{j,\varepsilon}$. Thus, the support of any solution h of (60) belongs to $\cup D_{j,\varepsilon}$. We shall identify functions f and h with vector functions whose components are the restrictions of f and h, respectively, to individual domains $D_{j,\varepsilon}$, $m+1 \leq j \leq N$. Furthermore, we map $D_{j,\varepsilon}$ onto ε-independent domain D_j by the transformation L'_ε defined by formulas

$$t - t_j = \varepsilon \widehat{t}, \qquad y = \varepsilon \widehat{y}. \tag{63}$$

This transformation differs from (44) by a shift in t (compare with (37)). The vector functions of variables $(\widehat{t}, \widehat{y})$ with components from $L^2(D_j)$ defined by f, h will be denoted by f' and h', respectively. Then equation (60) can be written in the form

$$h' + F'_\lambda h' = f',$$

where F'_λ is the $[(N-m)\times(N-m)]$-matrix operator which corresponds to the operator F_λ. Here $N-m$ is the number of finite channels in Ω_ε. Recall that the entries of the vectors f' and h' are functions with the domains D_j, which do not depend on ε (and λ). Our next goal is to describe how the entries of the matrix F'_λ depend on ε and λ. It will be done using (60), where each resolvent $R_{v,\lambda}$ can be specified using (47).

The first term in the right hand side of (47),

$$R_\lambda^{(s)} : L^2(C'_{s,\varepsilon}) \to L^2(C'_{s,\varepsilon}),$$

depends on ε. The transformation (63) maps $C'_{s,\varepsilon}$ onto the ε-independent cylinder C'_s. The operator

$$\widehat{R}_\lambda^{(s)} := L'_\varepsilon R_\lambda^{(s)} (L'_\varepsilon)^{-1} : L^2(C'_s) \to L^2(C'_s)$$

does not depend on ε (Lemma 12), and it depends meromorphically on λ. Thus, the contributions from the first term in the right hand side of (47) to the entries of the matrix F'_λ are operators which are ε-independent and meromorphic in λ. The rest of the terms in the right hand side of (47) (other than the remainder) are operators of finite ranks. Due to Lemma 13 (see also the formula (23) for $c_{s,k} = c^-_{s,k}$), the contributions of these terms to the entries of F'_λ are ε-independent operators which are analytic in λ and are of the rank one, multiplied by functions $q_{v;j,n,s,k}$ of the form

$$q_{v;j,n,s,k}(\lambda,\varepsilon) = e^{i\frac{\alpha_j\sqrt{\lambda-\lambda_{j,n}}+\beta_s\sqrt{\lambda-\lambda_{s,k}}}{\varepsilon}}. \tag{64}$$

Here $\alpha_j = t_j$ or $\alpha_j = l_j - t_j$ (independently, $\beta_s = t_s$ or $\beta_s = l_s - t_s$). Formula (47) leads to $\alpha_j = t_j$, $\beta_s = t_s$ if 1) the channels $C_{j,\varepsilon}$ and $C_{s,\varepsilon}$ are adjacent to a common junction, which corresponds to the vertex v, and 2) the parameter t on both channels $C_{j,\varepsilon}$ and $C_{s,\varepsilon}$ is introduced in such a way that $t=0$ at the vertex v. Other options in the choice of α_j and β_s correspond to opposite parametrization of the channels $C_{j,\varepsilon}$, $C_{s,\varepsilon}$, or both. If $C_{j,\varepsilon}$ and $C_{s,\varepsilon}$ do not have a common junction which corresponds to the vertex v then $q_{v;j,n,s,k}=0$. Thus, the matrix operator F'_λ can be represented in the form

$$F'_\lambda = F^0_\lambda + \left[\sum_{v;n,k} q_{v;j,n,s,k}(\lambda,\varepsilon) F_\lambda^{j,n,s,k}\right]_{j,s>m} + R, \tag{65}$$

where F^0_λ, $F_\lambda^{j,n,s,k}$ are ε-independent operators, F^0_λ is meromorphic in λ, operators $F_\lambda^{j,n,s,k}$ are analytic in λ and have rank one, the summation extends over all the verticis v and over $n \in [0, m_j]$,

$k \in [0, m_s]$. The operator $R = R(\varepsilon, \lambda)$ corresponds to the remainder term in (47) and has the following estimate

$$||R|| \le \frac{Ce^{-\gamma/\varepsilon}}{\Pi_{\lambda^j \in l}|\lambda - \lambda^j|}.$$

Since the analytic Fredholm theorem [2] is applicable to the operator $I + F'_\lambda$, from (65) it follows that it is also applicable to the operator $I + F^0_\lambda$. Let l^δ be the $\delta/2$-neighborhood of the set consisting of both the poles $\widehat{\lambda}^j$ of the operator $(I + F^0_\lambda)^{-1}$ located inside l and the points $\lambda^j \in l$. Then

$$||(I + F^0_\lambda)^{-1}|| \le C(\delta), \qquad ||R'|| \le C(\delta)e^{-\gamma/\varepsilon}, \qquad \lambda \in l \setminus l^\delta, \tag{66}$$

where

$$R' = R(I + F^0_\lambda)^{-1}.$$

Formula (65) implies, for $\lambda \in l \setminus l^\delta$,

$$(I + F'_\lambda)^{-1} = (I + F^0_\lambda)^{-1}\left[I + qG + R'\right]^{-1}, \tag{67}$$

where qG is the matrix operator with matrix elements $\sum_{v;n,k} q_{v;j,n,s,k}(\lambda, \varepsilon)G^{j,n,s,k}_\lambda$, $N - m < j, s \le N$. Here

$$G^{j,n,s,k}_\lambda = F^{j,n,s,k}_\lambda (I + F^0_\lambda)^{-1}.$$

The operators $G^{j,n,s,k}_\lambda$ are meromorphic in λ and have rank one.

The equation

$$(I + qG)x = g \tag{68}$$

for x can be reduced to an equation in the finite dimensional space S spanned by the ranges of the operators $G^{j,n,s,k}_\lambda$. We fix a basis in S, reduce equation (68) to the algebraic system $A\widehat{x} = \widehat{g}$ for coordinates of the projection of x on S, and solve the system using the Kramer rule. Since functions $q_{v;j,n,s,k}$ are bounded when $\lambda \in l$, the procedure described above allows us to estimate the norm of the operator $(I + qG)^{-1}$ through $|\det^{-1} A|$.

Hence, there exist a polynomial $P = P(q_{v;j,n,s,k})$ of variables $q_{v;j,n,s,k}$ which has the following properties. Its coefficients are meromorphic in λ (with poles belonging to the set $\{\widehat{\lambda}^j\} \cup \{\lambda^j\}$), the polynomial is linear with respect to each variable $q_{v;j,n,s,k}$, and is such that

$$||(I + qG)^{-1}|| \le C|f_\varepsilon(\lambda)|^{-1}, \qquad \lambda \in l \setminus l^\delta, \qquad f_\varepsilon(\lambda) := 1 + P(q_{v;j,n,s,k}(\lambda, \varepsilon)).$$

The function $f_\varepsilon(\lambda)$ here has the form (54) with one of g_j identically equal to zero, and the corresponding coefficient C_j equal to one. The latter implies (55). Thus, Lemma 15 can be applied to the function $f_\varepsilon(\lambda)$ above on each connected interval of the set $l \setminus l^\delta$. There are only finitely many such intervals. Thus, for any $\eta > 0$, there exists a thin set $\Gamma_\eta(\varepsilon)$ such that

$$||(I + qG)^{-1}|| \le Ce^{\eta/\varepsilon}, \qquad \lambda \in l \setminus \Gamma_\eta(\varepsilon).$$

We choose $\eta < \gamma$, where γ is defined in (66). Then

$$||(I + qG + R')^{-1}|| \le Ce^{\gamma/2\varepsilon}, \qquad \lambda \in l \setminus \Gamma_\eta(\varepsilon),$$

when ε is small enough,. A similar estimates holds for operator (67):

$$||(I + F'_\lambda)^{-1}|| \le Ce^{\gamma/2\varepsilon}, \qquad \lambda \in l \setminus \Gamma_\eta(\varepsilon).$$

Hence, the same estimate is valid for the operator $(I + F_\lambda)^{-1}$, and from (59), (61) it follows that

$$R_\lambda f = \sum_{v \in V} \psi_v R_{v,\lambda}(\phi_v h), \qquad \lambda \in l\ , \tag{69}$$

where $h \in L^2(\Omega_\varepsilon)$ is supported in $\cup D_{j,\varepsilon}$, $j > m$, and

$$||h||_{L^2(\Omega_\varepsilon)} \leq Ce^{\gamma/2\varepsilon}||f||_{L^2(\Omega_\varepsilon)}, \quad \lambda \in l \setminus \Gamma_\eta(\varepsilon). \tag{70}$$

Relations (69), (70), (46), and Lemma 11 together provide the asymptotic expansion (15) for $R_\lambda f$ needed to complete the proof of Theorem 6.

The last result, which we are going to discuss now, concerns the limiting behavior of the GC as λ approaches λ_0, the bottom of the absolutely continuous spectrum. We assume that the resolvent (12) does not have a pole at $k = \sqrt{\lambda_0}$. Obviously this assumption holds for generic domains Ω_ε. Theorem 4 implies that this assumption is equivalent to the absence of bounded solutions of the homogeneous problem (5) with $\lambda = \lambda_0$. Recall that the scattering matrix (10) depends on $\lambda > \lambda_0$ and the GC (18) depend on both $\lambda > \lambda_0$ and $\varepsilon > 0$.

Proof of Theorem 7. Consider an infinite channel C_s, for which $\lambda_{s,0} = \lambda_0$. Let $\Psi_{s,0}^{(\varepsilon)}$ be the scattering solution which corresponds to the incident wave

$$\psi_{inc} = e^{-i\frac{\sqrt{\lambda-\lambda_{s,0}}}{\varepsilon}t}\varphi_{s,0}(y/\varepsilon).$$

Let $\phi_s \in C^\infty(\Omega_\varepsilon)$, $\phi_s = 1$ in the channel C_s for $t \geq 2$, $\phi_s = 0$ in C_s for $t \leq 1$ and outside of C_s. We represent $\Psi_{s,0}^{(\varepsilon)}$ in the form

$$\Psi_{s,0}^{(\varepsilon)} = \phi_s\psi_{inc} + u, \quad \lambda > \lambda_0.$$

Then u is the outgoing solution of the problem

$$(-\varepsilon^2\Delta - \lambda)u = f, \quad x \in \Omega_\varepsilon; \qquad u = 0 \ \text{ on } \partial\Omega_\varepsilon,$$

where $f = -\varepsilon^2(\Delta\phi_s)\psi_{inc} - 2\varepsilon^2\nabla\phi_s\nabla\psi_{inc}$ has a compact support. Hence, $u = R_\lambda f$. From here, the second statement of Theorem 4, and the absence of a pole at $k = \sqrt{\lambda_0}$ it follows that u, if considered as an element of $L^2_{loc}(\Omega_\varepsilon)$, is analytic in $z = \sqrt{\lambda - \lambda_0}$ in a neighborhood of the point $z = 0$. Then from standard local a priory estimates for solutions of elliptic problems it follows that u is analytic, if considered as an element of any Sobolev space of functions on any bounded part of Ω_ε. Hence, the restrictions u_j of u to cross-sections $t = 2$ of infinite channels C_j are analytic in z. Thus, for any infinite channel C_j, u is an outgoing solution of the problem

$$(-\varepsilon^2\Delta - \lambda)u = 0, \quad x \in \Omega_\varepsilon \cap \{t > 2\}; \quad u = 0 \ \text{ on } \partial\Omega_\varepsilon \cap \{t > 2\}; \quad u|_{t=2} = u_j. \tag{71}$$

Since u_j is analytic in $z = \sqrt{\lambda - \lambda_0}$ in a neighborhood of the point $z = 0$, the coefficients $a_{j,n}$ in the asymptotic expansion (6) for the solution u of (71) are analytic in z. This proves the analyticity of the scattering matrix.

From analyticity of u in z and (71) it also follows that the scattering solution $\Psi_{s,0}^{(\varepsilon)}$, when $z = 0$, is a solution of the homogeneous problem (5) with $\lambda = \lambda_0$, and satisfies (13). Thus $\Psi_{s,0}^{(\varepsilon)} \equiv 0$ when $z = 0$ due to Theorem 4. This implies that $T = -I$ and completes the proof of the first statement. The second statement of the theorem is an obvious consequence of the analyticity of T_v and (18). This completes the proof of the theorem.

Remarks concerning Theorem 7. 1) Consider a bounded domain Ω_ε with one junction and several channels of finite length. Let Ω'_ε be a spider type domain which one gets by extending the channels of Ω_ε to infinity. The spectrum of the problem (5) in Ω_ε is discrete, and there exists a sequence of eigenvalues which approach λ_0 as $\varepsilon \to 0$. Each of these eigenvalues has the form

$$\lambda_n(\varepsilon) = \lambda_0 + O(\varepsilon^2). \tag{72}$$

Theorem 7, concerning the problem in Ω'_ε, can be used to specify the asymptotic behavior (72) of the eigenvalues $\lambda_n(\varepsilon)$. The last statement of the theorem and (72) indicate that, for generic domains

Ω_ε, the asymptotic behavior of $\lambda_n(\varepsilon)$ as $\varepsilon \to 0$ (when n is fixed or $n \to \infty$ not very fast) is the same as for eigenvalues of the corresponding Dirichlet problem on the limiting graph with the Dirichlet GC at the vertex. This result will be discussed in more detail elsewhere.

2) Our paper [14] contains a mistake in the statement of Theorem 5.1 (which is a simplified version of Theorem 7 above) about the form of the GC at the bottom of the absolutely continuous spectrum: $k \to 0$ has to be replaced there by $k \to 0$, $\varepsilon \to 0$. The arguments in the last 5 lines of the proof are wrong, but can be easily corrected with the additional assumption that $\varepsilon \to 0$. (Also, the index of summation in the formulas (5.2), (5.4), (5.6) of that paper must be n, not j).

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